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1993 J. Phys. A: Math. Gen. 26 1249

(http://iopscience.iop.org/0305-4470/26/5/040)

J. Phys. A: Math. Gen. 26 (1993) 1249–1257. Printed in the UK

Exact solution of a one-dimensional model of hole superconductivity*

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Received 4 September 1992, in final form 7 December 1992

Abstract. A new integrable model of a strongly correlated electronic system is formulated as a model of hole superconductivity. The model is solved by using the Bethe ansatz. The critical exponents describing the decrease of correlation functions on long distances are derived. The behaviour of these correlations indicates that Cooper pairs of holes are formed in the repulsive region of the model. This conclusion is also confirmed by the calculation of the conductivity and the effective transport mass. In the attractive region, the model is a highly conducting system in which the current carriers with small effective mass are 'light fermions'.

1. Introduction

The discovery of high-temperature superconductivity [1] has greatly stimulated the interest in low-dimensional electronic systems with strong correlations. Among the relevant models the one-dimensional Hubbard [2] and the (supersymmetric) t-J models [3–7] are special since they can be treated exactly in terms of the Bethe ansatz. As exact results are highly desirable, particularly for low-dimensional systems in the strong correlation regime, these models have been extensively studied. The physical motivation for considering the Hubbard and the t-J model is the fact that the motion of electrons is strongly influenced by the on-site Coulomb repulsion and by the spin fluctuations through antiferromagnetic coupling, respectively.

Another approach to high-temperature superconductivity proposed by Hirsch [8, 9] makes it possible to formulate a new integrable model of strongly correlated systems. According to [9] the charge carriers of high-temperature superconductors are holes. The kinetic energy of hole hopping between nearest-neighbour sites depends on the occupation of these sites. In such a model the repulsive Coulomb interaction for electrons leads to an attractive interaction for holes which is strongest at low densities of holes.

In the following we shall consider a simplified version of Hirsch's model on a one-dimensional chain of even length L, closed periodically. It is given by the

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Hamiltonian

$$H(t) = -\sum_{j=1}^{L}\sum_{\sigma=\pm 1}(c_{j\sigma}^{+}c_{j+1\sigma} + c_{j+1\sigma}^{+}c_{j\sigma})(1 + t\, n_{j+(1+\sigma)/2,-\sigma}) \tag{1}$$

with interaction parameter t, where $c_{j\sigma}$ is the annihilation operator for an electron with spin σ at site j and $n_{j\sigma} = c_{j\sigma}^{+}c_{j\sigma}$. For comparison we present the Hamiltonian $\bar{H}$ of the original Hirsch model [9]

$$\bar{H} = -\sum_{j=1}^{L}\sum_{\sigma=\pm 1}(c_{j\sigma}^{+}c_{j+1\sigma} + c_{j+1\sigma}^{+}c_{j\sigma})\left[1 + \tfrac{1}{2}t(n_{j+(1+\sigma)/2,-\sigma} + n_{j+(1-\sigma)/2,-\sigma})\right]. \tag{2}$$

Obviously our Hamiltonian (1) only contains half of the interaction terms of $\bar{H}$. The advantage of model (1) is that it is solvable by the Bethe ansatz whereas model (2) is not, as a direct calculation shows that its S-matrix does not satisfy the Yang–Baxter equations [10, 11]. These are satisfied for $\bar{H}$ only in the continuum limit, i.e. in the limit of low or high densities of particles. The last case is the most important one from the physical point of view, because here the existence of hole pairs of Cooper type can be expected. A direct comparison of the S-matrices of Hamiltonians (1) and (2) shows that the continuum versions of both models coincide. Therefore it can be expected that both models have identical critical properties. Nevertheless a proof of this claim, based for instance on the renormalization group approach, is highly desirable.

In any case, model (1) keeps the main idea of Hirsch's approach to superconductivity, namely the modulation of the hopping process by the presence of other particles as the main reason for superconductivity. Hamiltonian (1) includes the simplest form of terms describing such processes. The various arguments to support the possibility that such terms could overwhelm the direct Coulomb repulsion and may lead to superconductivity have been given in [8, 9]. This is reason enough to study the superconducting properties of model (1). In [12] another model for interacting fermions including correlated hopping terms was constructed. The ground state of this system can be given explicitly in any dimension, in one dimension the system is integrable. The model of [12] and (1) are related, the latter one however enjoys a simpler physical interpretation.

In [13] one of the authors has shown that model (1) is solvable by using the Bethe ansatz. The relevant equations have been derived and the ground-state energy has been calculated. In the present paper we shall mainly consider the correlation functions of the model in order to investigate the possibility of superconductivity.

As a model for electrons the interaction term in (1) should be negative, i.e. $t < 0$, corresponding to Coulomb repulsion. If, according to [9] this leads to an attraction of holes, this fact must be seen in the behaviour of the hole correlation functions. The corresponding Hamiltonian for holes is obtained from (1) by a particle-hole transformation U together with a sublattice rotation [14], namely $c_{j\sigma}^{+} \Rightarrow (-1)^{j} c_{j\sigma}$. Multiplying also with a suitable scale factor, $(1+t)^{-1}$, the resulting hole Hamiltonian $H(t')$

$$H(t') = (1+t)^{-1}UH(t)U^{-1} \qquad t' = -\frac{t}{1+t} \tag{3}$$

is then of the same form as (1). Now, however, the t'-interaction is positive. As a consequence we shall study model (1) with attractive interaction, i.e. with $t > 0$, regarding its correlation functions as hole correlation functions of the repulsive model.

2. The Bethe ansatz

The Bethe ansatz for the model has been formulated in [13], from which we quote the relevant equations. The energy eigenstates are characterized by sets of wave numbers k_j for the particles and additional parameters Λ_α. Each of the latter ones is related to a particle with down spin. The Bethe ansatz wave numbers k_j and Λ_α satisfy a set of nonlinear equations derived in [13]

$$\begin{aligned} Lk_j &= 2\pi I_j + \sum_{\beta=1}^{M} \Theta\left(k_j - \Lambda_\beta; \frac{\eta}{2}\right) \qquad j = 1, \dots, N \\ \sum_{j=1}^{N} \Theta\left(\Lambda_\alpha - k_j; \frac{\eta}{2}\right) &- \sum_{\beta=1}^{M} \Theta(\Lambda_\alpha - \Lambda_\beta; \eta) = 2\pi J_\alpha \qquad \alpha = 1, \dots, M \end{aligned} \tag{4}$$

with the phase shift function

$$\Theta(k; \eta) = 2 \tan^{-1}\left(\coth \eta \tan \tfrac{1}{2}k\right) \qquad -\pi \leqslant \Theta < \pi \tag{5}$$

and the interaction parameter $\eta = \ln(1 + t)$. Furthermore N is the total number of particles, M is the number of particles with down spin, and I_j and J_α are integers or half-odd integers depending on the parities of N and M. The energy and momentum of the corresponding state are given by

$$\begin{aligned} E &= -\sum_{j=1}^{N} 2\cos k_j + \mu N \\ P &= \sum_{j=1}^{N} k_j = \frac{2\pi}{L}\left(\sum_{j=1}^{N} I_j + \sum_{\alpha=1}^{M} J_\alpha\right) \end{aligned} \tag{6}$$

where, from now on, the chemical potential μ has been added to control the particle number.

Equations (4) and (6) hold regardless of the sign of η, nevertheless the structure of the solutions is very different for $\eta < 0$ and $\eta > 0$. In [13] the model was considered for $\eta < 0$. Here we treat (4) for positive η in the symmetric case when there are equally many particles with spin up and spin down ($M = N/2$). The eigenstates consist of a certain number of singlet bound pairs and a certain number of free particles. The bound pairs are characterized by pairs of complex wavenumbers $k^\pm$

$$k_\alpha^\pm = \Lambda_\alpha \pm \mathrm{i}\eta. \tag{7}$$

In the ground state we only have pairs. Using (7) the above equations are reduced to the following set of equations after some simple algebra

$$2L\Lambda_\alpha = 2\pi J_\alpha + \sum_{\beta=1}^{M} \Theta(\Lambda_\alpha - \Lambda_\beta; \eta)$$

$$E = \sum_{\alpha=1}^{M} \epsilon_0(\Lambda_\alpha) \qquad \epsilon_0(\Lambda) = 2\mu - 4\cosh\eta\cos\Lambda \tag{8}$$

$$P = 2\sum_{\alpha=1}^{M} \Lambda_\alpha .$$

The ground state is characterized by the following values of J_α

$$J_\alpha^0 = \alpha - (M+1)/2 \qquad (\alpha = 1, 2, \ldots, M). \tag{9}$$

Deviations from this distribution of J_α describe gapless excitations of particle-hole type. In the thermodynamic limit $L \to \infty$, $M \to \infty$ for fixed ratio M/L the values of Λ_α fill an interval $[-\Lambda_0, \Lambda_0]$ uniformly with density $\sigma(\Lambda)$. From (8) we then obtain the integral equation for the distribution function $\sigma(\Lambda)$

$$2\pi\sigma(\Lambda) + \int_{-\Lambda_0}^{\Lambda_0} \Theta'(\Lambda - \widetilde{\Lambda}; \eta)\sigma(\widetilde{\Lambda})\,\mathrm{d}\widetilde{\Lambda} = 2 \qquad \Theta'(\Lambda, \eta) = \frac{\sinh 2\eta}{\cosh 2\eta - \cos\Lambda} \tag{10}$$

with the subsidiary condition

$$\int_{-\Lambda_0}^{\Lambda_0} \sigma(\Lambda)\,\mathrm{d}\Lambda = \rho \tag{11}$$

where $2\rho = 2M/L$ is the density of the hole liquid. For fixed chemical potential the parameter Λ_0 must be chosen to minimize the ground-state energy, given by

$$\begin{aligned} E_0/L &= \int_{-\Lambda_0}^{\Lambda_0} \epsilon_0(\Lambda)\sigma(\Lambda)\,\mathrm{d}\Lambda \\ &= \frac{1}{\pi}\int_{-\Lambda_0}^{\Lambda_0} \epsilon(\Lambda)\,\mathrm{d}\Lambda \end{aligned} \tag{12}$$

where in the second representation the dressed energy $\epsilon(\Lambda)$ has been used which is the solution of the integral equation

$$\epsilon(\Lambda) + \frac{1}{2\pi}\int_{-\Lambda_0}^{\Lambda_0} \Theta'(\Lambda - \widetilde{\Lambda}; \eta)\epsilon(\widetilde{\Lambda})\,\mathrm{d}\widetilde{\Lambda} = \epsilon_0(\Lambda) \tag{13}$$

such that $\epsilon(\pm\Lambda_0) = 0$ which is the minimization condition. The solution of (13) also defines the energy of particle-hole excitations. The momentum of such an excitation is given by

$$P(\Lambda) = 2\Lambda - \int_{-\Lambda_0}^{\Lambda_0} \Theta(\Lambda - \widetilde{\Lambda}; \eta)\sigma(\widetilde{\Lambda})\,\mathrm{d}\widetilde{\Lambda}. \tag{14}$$

The only other type of excitation consists of broken pairs. The breaking-up of one bound pair leads to the creation of two free particles. The energies and momenta of these particles are obtained from (4) and (6), for instance in the case when there are $M-1$ bound pairs and two particles with real momenta $k_{1,2}$. The energy of each free particle with momentum k is given by

$$\epsilon_f(k) = \mu - 2\cos k - \frac{1}{2\pi}\int_{-\Lambda_0}^{\Lambda_0} \Theta'\left(\Lambda - k; \frac{\eta}{2}\right)\epsilon(\Lambda)\,d\Lambda. \tag{15}$$

We content ourselves with pointing out that this excitation has a gap at $k=0$. A detailed study of this type of excitations for a more general model will be presented in [15].

In order to study the ground-state correlations of the model, we use two different approaches. First, following [16, 17], we calculate the critical exponents and determine the long-distance behaviour of the two-point correlation functions. Secondly, we investigate the conductivity of the model as a function of the particle density. This approach is based on the calculation of the ground-state energy under twisted boundary conditions [18, 19].

3. Critical exponents of the correlation functions

To obtain the critical exponents of the correlation functions we use the predictions of conformal field theory [20, 21]. According to this theory there is a one-to-one correspondence between the conformal dimensions of the scaling operators and the finite-size corrections to the energy of the excited states of the critical Hamiltonian. Our model is critical since the gapless excitations have a linear dispersion law in the vicinity of the Fermi points. The excitations corresponding to the breaking-up of bound pairs have a gap as mentioned before. These excitations do not affect the critical properties and the finite-size corrections. The finite-size corrections to the gapless excitations (11) can be calculated in a straightforward way [22, 23]. Omitting the details of the calculation, we only present the results.

We denote the change of the number of bound pairs as ΔM and the number of pairs moved from the left to the right Fermi point as d. The $1/L$-corrections to the low-energy excitations are then

$$\Delta E = \frac{2\pi v_F}{L}\left(\frac{(\Delta M)^2}{[2\xi(\Lambda_0)]^2} + [\xi(\Lambda_0)]^2 d^2 + N^+ + N^-\right) \tag{16}$$

where $\xi(\Lambda_0)$ is the dressed charge [24] at the Fermi surface, the dressed charge function $\xi(\Lambda)$ being defined through the modified integral equation (9)

$$\xi(\Lambda) + \frac{1}{2\pi}\int_{-\Lambda_0}^{\Lambda_0} \Theta'(\Lambda - \widetilde{\Lambda}; \eta)\xi(\widetilde{\Lambda})\,d\widetilde{\Lambda} = 1 \tag{17}$$

such that $\xi \equiv \pi\sigma$ in the present case. v_F is the Fermi velocity

$$v_F = \frac{\epsilon'(\Lambda_0)}{2\pi\sigma(\Lambda_0)} = \frac{\epsilon'(\Lambda_0)}{2\xi(\Lambda_0)}. \tag{18}$$

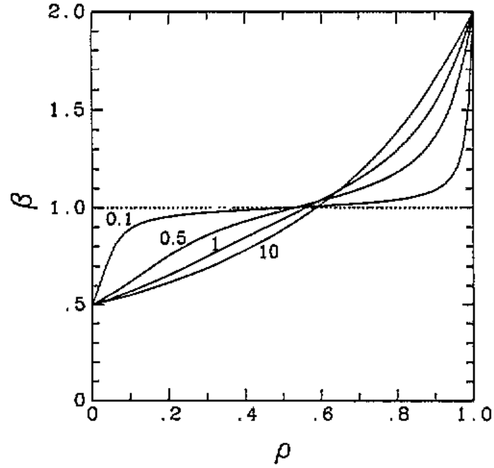

Figure 1. Depiction of the exponent β of the pair correlation function for different values of the interaction parameter $\eta = 0.1, 0.5, 1, 10$.

The non-negative integers $N^{\pm}$ are quantum numbers of the simple particle-hole excitations. The momentum associated with these excitations is

$$P = 2k_{\mathrm{F}}d + \frac{2\pi}{L}\left(d\Delta M + N^{+} - N^{-}\right) \qquad k_{\mathrm{F}} = \pi\frac{M}{L}. \tag{19}$$

To read off the conformal dimensions $\Delta_{\pm}$ from these expressions one has to compare (16) and (19) with the predictions of conformal field theory [25, 26]. Then we have

$$\Delta_{\pm} = \frac{1}{2}\left(\frac{\Delta M}{2\xi(\Lambda_0)} \pm \xi(\Lambda_0)d\right)^2 + N^{\pm}. \tag{20}$$

We notice that in contrast to the repulsive case [27] this formula is valid for integer d without selection rule for fermions, since in the present case it corresponds to bound pairs. It is a reason why we consider the particle correlation function of the model with an attractive interaction as a hole correlation function in the repulsive case.

Now we consider the long-distance behaviour of correlation functions. The asymptotic form of the density correlation function is given by

$$\langle\rho(r)\rho(0)\rangle \simeq \rho^2 + A_1 r^{-2} + A_2 r^{-\alpha}\cos(2k_{\mathrm{F}}r). \tag{21}$$

The non-oscillating part arises from the lowest particle-hole excitations. The relevant excitation for the $2k_{\mathrm{F}}$ oscillation term is $(\Delta M, d, N^{\pm}) = (0, 1, 0)$. We thus find the critical exponent

$$\alpha = 2(\Delta_{+} + \Delta_{-}) = 2[\xi(\Lambda_0)]^2. \tag{22}$$

The excitations relevant for the correlation function of singlet pairs are specified by $(\Delta M, d,\ N^{\pm}) = (1, 0, 0)$. We then obtain the asymptotic behaviour of this correlation

$$G_p(r) = \langle c^{+}_{r\uparrow} c^{+}_{r\downarrow} c_{0\downarrow} c_{0\uparrow}\rangle \simeq B r^{-\beta} \tag{23}$$

where

$$\beta = \frac{1}{2[\xi(\Lambda_0)]^2} = \alpha^{-1}. \tag{24}$$

The exponent β is plotted in figure 1 for some values of the interaction parameter η by numerically solving equations (10), (11) or (17), respectively, from which $\xi(\Lambda_0)$ is determined.

Let us discuss the results. In one-dimensional systems we have no superconductivity in the literal sense. However, the power-decay of the singlet pair correlation (23) indicates the existence of singlet pairs provided that the exponent β of this correlation is smaller than that of the density–density correlation α [16]. In this case the correlation of singlet pairs overwhelms the density correlation, and we can say that the particles are confined in pairs. From figure 1 we see that such behaviour always exists for particle concentrations $\rho \leqslant \rho_c$. The critical concentration ρ_c is defined by $\beta(\rho_c) = 1$ and varies monotonically from 1/2 to $2-\sqrt{2} = 0.5858\ldots$ with increasing interaction η.

We remark that these results for the model with attractive interaction can be applied to the model with repulsive interaction by a particle-hole transformation. Therefore we have Cooper type singlet pairs of holes in the model of repulsive electrons.

4. Conductivity and effective transport mass

In order to substantiate the physical picture given above we now study the conductivity and the effective transport masses which can be calculated following the ideas of [18, 19]. To this end we change the periodic boundary condition leading to (8) by a twisted one with twisting angle φ. In this case instead of (8) we obtain

$$2L\Lambda_\alpha = 2\pi I_\alpha + 2\varphi + \sum_{\beta=1}^{M} \Theta(\Lambda_\alpha - \Lambda_\beta; \eta). \tag{25}$$

Physically the additional phase φ can be obtained by enclosing a magnetic flux in the ring on which the electrons can move. For small φ this leads to a change in the ground-state energy

$$\Delta E_0 = D_c \varphi^2 / L \tag{26}$$

where D_c is the charge stiffness. The conductivity of the system is directly proportional to D_c [18, 19]. In order to see the correlation effects clearly it is useful to introduce the effective transport mass m defined by the relation

$$\frac{m}{m_e} = \frac{D_c^0}{D_c} \tag{27}$$

where $D_c^0 = \frac{2}{\pi}\sin(\pi\rho)$ is the charge stiffness of the non-interacting system and m_e is the electron mass. On the other hand this change in the boundary conditions corresponds to the finite-size correction (16) for $\Delta M = 0$, $d = \varphi/\pi$ and we have for the charge stiffness of the Hamiltonian (1) with attractive interactions ($t > 0$)

$$D_c^{>} = \frac{2}{\pi} v_F \xi^2(\Lambda_0) = \frac{1}{\pi}\epsilon'(\Lambda_0)\xi(\Lambda_0) \tag{28}$$

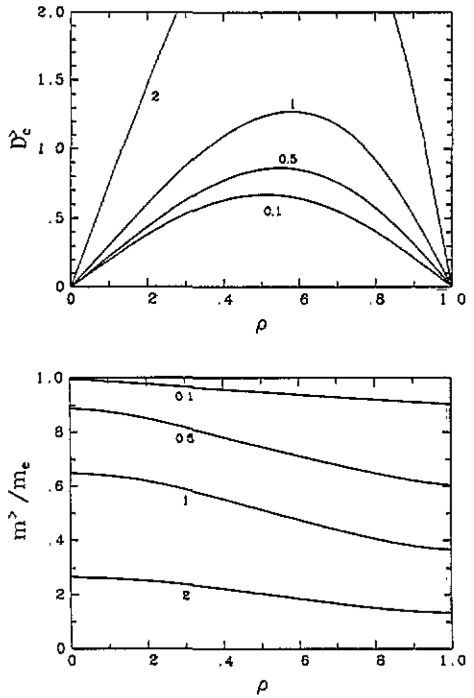

Figure 2. Dependence of charge stiffness $D_c^>$ and effective transport mass $m^>$ on ρ for different interactions $\eta = 0.1$, 0.5, 1, 2. Note that ρ is the density of particles in the attractive model.

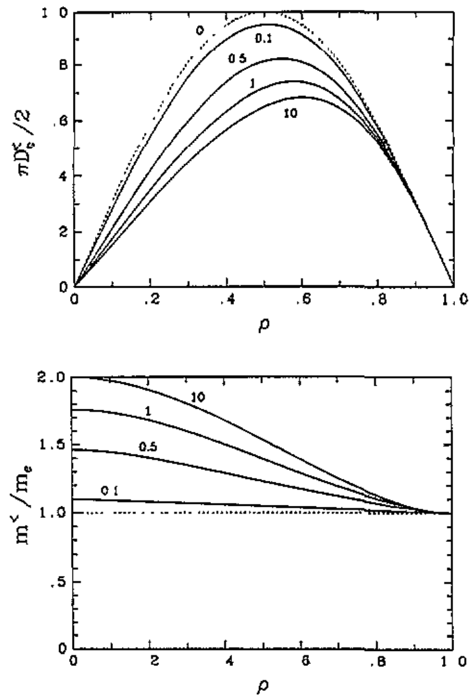

Figure 3. Depiction of charge stiffness $D_c^<$ and effective transport mass $m^<$ after rescaling using (29). Note that in this repulsive case ρ is the density of holes.

and with repulsive interactions ($t < 0$)

$$D_c^< = e^{-\eta} D_c^> = \frac{2}{\pi} v_F \xi^2(\Lambda_0) e^{-\eta}. \tag{29}$$

The result (29) can also be obtained by using results of [27] where the repulsive model has been investigated directly. In figures 2 and 3 we present the conductivity and effective transport mass as a function of the particle concentration for the Hamiltonian (1) with attractive and repulsive interactions, respectively.

We begin with the discussion of the conductivity of the repulsive model. From figure 3 it is clear that the conductivity in the high-density limit vanishes linearly as the concentration decreases ($D_c^< \sim 1 - \rho$). This is simply due to the decrease of the carrier density just as for the non-interacting case. It indicates that in this region the current carriers are the free electrons of the repulsive model. It is noteworthy that in the low-density limit the effective mass is enhanced by a factor of two. This behaviour and our previous findings for the correlation functions can be interpreted as the formation of hole pairs due to an attractive force between holes.

In the high-density limit of the attractive model we also observe a linear decrease $D_c^> \sim 1 - \rho$ (figure 2). In this case the free particles which carry the current are holes. For all densities ρ and interactions η the effective masses are reduced in comparison to the non-interacting case. Furthermore the masses decrease with increasing interaction parameter t ($m^{>}/m_e \sim t^{-1}$). As the effective mass of the current carriers become very small, one may call them 'light fermions'.

Acknowledgments

One of the authors (RZB) gratefully acknowledges the hospitality of the Institut für Theoretische Physik der Universität zu Köln as well as partial financial support by the Scientific Council on High Temperature Superconductivity (Russia) under grant No 91-151. He also thanks Professor D I Khomsky for discussions.

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